

FOURIER MULTIPLIERS ON GRADED LIE GROUPS

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ABSTRACT. We study the L^p -boundedness of Fourier multipliers defined on graded nilpotent Lie groups via their group Fourier transform. We show that Hörmander type conditions on the Fourier multipliers imply L^p -boundedness. We express these conditions using difference operators and positive Rockland operators. We also obtain a more refined condition using Sobolev spaces on the dual of the group which are defined and studied in this paper.

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1. INTRODUCTION

The Mihlin multiplier theorem [13, 14] states that if a function σ defined on $\mathbb{R}^n \setminus \{0\}$ has at least $[d/2] + 1$ continuous derivatives that satisfy

$$\forall \alpha \in \mathbb{N}_0^d, |\alpha| \leq [d/2] + 1, \quad |\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad (1.1)$$

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(where $[t]$ is the integer part of t), then the Fourier multiplier operator T_σ associated with σ , initially defined on Schwartz functions via

$$T_\sigma \phi := \mathcal{F}^{-1} \{ \sigma \widehat{\phi} \}, \quad (1.2)$$

(where $\mathcal{F}\phi = \widehat{\phi}$ denotes the Euclidean Fourier transform) admits a bounded extension on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. Hörmander improved the Mihlin multiplier theorem by showing [10] that a sufficient condition for T_σ to be bounded on $L^p(\mathbb{R}^d)$ is the membership of σ locally uniformly to a Sobolev space $H^s(\mathbb{R}^d)$ for some $s > d/2$, that is,

$$\exists \eta \in \mathcal{D}(0, \infty), \eta \neq 0, \quad \sup_{r>0} \|\sigma(r \cdot) \eta(| \cdot |^2)\|_{H^s} < \infty. \quad (1.3)$$

The Hörmander condition in (1.3) with s near enough $d/2$ implies Mihlin's condition in (1.1). Anisotropic analogues of the Hörmander condition in (1.3) have been studied by Rivière [16].

In this paper, we present analogues of the Hörmander and Mihlin conditions in the context of Lie groups endowed with (anisotropic) dilations, and show that they imply the L^p -boundedness of the corresponding Fourier multiplier operators. In the context of (type 1) Lie groups, the Fourier multipliers are defined formally as in (1.2) but replacing the Euclidean Fourier transform with the group Fourier transform. A multiplier symbol σ is now a field of operators parametrised by the dual \widehat{G} of the group G . Any two multiplier symbols may not necessarily commute.

We will give the analogue of the Hörmander-type condition in Theorem 4.11, after having defined and studied Sobolev spaces on \widehat{G} . We can already give the analogue of the Mihlin-type condition (which, we will see, is implied by the Hörmander-type condition of Theorem 4.11):

Theorem 1.1. *Let G be a graded nilpotent Lie group. Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators in $L^2(\widehat{G})$. If there exist a positive Rockland operator \mathcal{R} such that for any $|\alpha| \leq N$ the following quantities are finite:*

$$\sup_{\pi \in \widehat{G}} \|\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma\|_{\mathcal{L}(\mathcal{H}_\pi)} \quad \text{and} \quad \sup_{\pi \in \widehat{G}} \|\Delta^\alpha \sigma \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)}, \quad (1.4)$$

where ν is the degree of \mathcal{R} and N is the smallest integer strictly greater than half the homogeneous dimension and divisible by the dilation weights v_1, \dots, v_n ,

$$\text{i.e. } \frac{N}{v_j} \in \mathbb{N}, \quad j = 1, \dots, n, \quad \text{and} \quad N > \frac{Q}{2} = \frac{v_1 + \dots + v_n}{2},$$

then the Fourier multiplier operator T_σ corresponding to σ is bounded on $L^p(G)$ for any $1 < p < \infty$. Furthermore,

$$\|T_\sigma\|_{\mathcal{L}(L^p(G))} \leq C \sum_{|\alpha| \leq N} \left(\sup_{\pi \in \widehat{G}} \|\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma\|_{\mathcal{L}(\mathcal{H}_\pi)} + \sup_{\pi \in \widehat{G}} \|\Delta^\alpha \sigma \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)} \right),$$

with $C = C_{p,G}$ independent of σ .

The precise definitions of graded homogeneous Lie groups, their dilations weights and Rockland operators amongst others will be recalled in Section 2.

Theorem 1.1 applied to the abelian Euclidean setting, that is, $(\mathbb{R}^d, +)$ with the usual isotropic dilation with \mathcal{R} being the Laplace operator, boils down to the Mihlin theorem. It will also be the case for Theorem 4.11. Indeed in the Euclidean abelian setting, $\pi(\mathcal{R})$ is replaced with $|\xi|^2$ where ξ is the (Fourier) dual variable.

The L^p -multiplier problem has been extensively studied in various contexts. On Lie groups, these studies have focussed on spectral multipliers of one (or several) operator such as a sub-Laplacian. The only exceptions known to the authors are on compact Lie groups, first on $SU(2)$ by Coifman and Weiss in 1971, see [1], and more recently on any compact Lie group by Ruzhansky and Wirth [17]. As in the latter paper, our approach uses difference operators. We will also use the Calderón-Zygmund theory adapted to the setting of spaces of homogeneous type as in [2], see also [16].

The results of this paper may be applied to show that the Riesz operators associated with Rockland operators are bounded on some L^p spaces, see Remark 4.13.

The paper is organised as follows. In Section 2, we recall the necessary material regarding the setting. In Section 3, we define and study Sobolev spaces on \widehat{G} . In Section 4, we present our Mihlin-Hörmander condition. In Section 5, we show the statements of the previous section.

Notation: $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of non-negative integers and $\mathbb{N} = \{1, 2, \dots\}$ the set of positive integers. If \mathcal{H}_1 and \mathcal{H}_2 are two Hilbert spaces, we denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of the bounded operators from \mathcal{H}_1 to \mathcal{H}_2 . If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then we write $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{L}(\mathcal{H})$. We may allow ourselves to write $A \asymp B$ when the quantity A and B are equivalent in the sense that there exists a constant such that $C^{-1}A \leq B \leq CA$.

2. PRELIMINARIES

In this section, after defining graded Lie groups, we recall their homogeneous structure, some general representation theory in this context as well as the definition and some properties of their Rockland operators.

2.1. Graded and homogeneous Lie groups. Here we recall briefly the definition of graded nilpotent Lie groups and their natural homogeneous structure. A complete description of the notions of graded and homogeneous nilpotent Lie groups may be found in [8, ch1] and [7].

We will be concerned with graded Lie groups G which means that G is a connected and simply connected Lie group whose Lie algebra \mathfrak{g} admits an \mathbb{N} -gradation $\mathfrak{g} = \bigoplus_{\ell=1}^{\infty} \mathfrak{g}_{\ell}$ where the \mathfrak{g}_{ℓ} , $\ell = 1, 2, \dots$, are vector subspaces of \mathfrak{g} , almost all equal to $\{0\}$, and satisfying $[\mathfrak{g}_{\ell}, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'}$ for any $\ell, \ell' \in \mathbb{N}$. This implies that the group G is nilpotent. Examples of such groups are the Heisenberg group and, more generally, all stratified groups (which by definition correspond to the case \mathfrak{g}_1 generating the full Lie algebra \mathfrak{g}).

We construct a basis X_1, \dots, X_n of \mathfrak{g} adapted to the gradation, by choosing a basis $\{X_1, \dots, X_{n_1}\}$ of \mathfrak{g}_1 (this basis is possibly reduced to $\{0\}$), then $\{X_{n_1+1}, \dots, X_{n_1+n_2}\}$ a basis of \mathfrak{g}_2 (possibly $\{0\}$ as well as the others) and so on. Via the exponential mapping $\exp_G : \mathfrak{g} \rightarrow G$, we identify the points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with the points $x = \exp_G(x_1 X_1 + \dots + x_n X_n)$ in G . Consequently we allow ourselves to denote

by $C(G)$, $\mathcal{D}(G)$ and $\mathcal{S}(G)$ etc, the spaces of continuous functions, of smooth and compactly supported functions or of Schwartz functions on G identified with \mathbb{R}^n , and similarly for distributions with the duality notation $\langle \cdot, \cdot \rangle$.

This basis also leads to a corresponding Lebesgue measure on \mathfrak{g} and the Haar measure dx on the group G , hence $L^p(G) \cong L^p(\mathbb{R}^n)$. The group convolution of two functions f and g , for instance integrable, is defined via

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)dy.$$

The convolution is not commutative: in general, $f * g \neq g * f$, but the Young convolutions inequalities hold, so that we have

$$\|f * g\|_{L^r(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^q(G)}, \quad p, q, r \in [1, \infty], \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (2.1)$$

The coordinate function $x = (x_1, \dots, x_n) \in G \mapsto x_j \in \mathbb{R}$ is denoted by x_j . More generally we define for every multi-index $\alpha \in \mathbb{N}_0^n$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, as a function on G . Similarly we set $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$ in the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ of \mathfrak{g} .

For any $r > 0$, we define the linear mapping $D_r : \mathfrak{g} \rightarrow \mathfrak{g}$ by $D_r X = r^\ell X$ for every $X \in \mathfrak{g}_\ell$, $\ell \in \mathbb{N}$. Then the Lie algebra \mathfrak{g} is endowed with the family of dilations $\{D_r, r > 0\}$ and becomes a homogeneous Lie algebra in the sense of [8]. We rewrite the set of integers $\ell \in \mathbb{N}$ such that $\mathfrak{g}_\ell \neq \{0\}$ into the increasing sequence of positive integers v_1, \dots, v_n counted with multiplicity, the multiplicity of \mathfrak{g}_ℓ being its dimension. In this way, the integers v_1, \dots, v_n become the weights of the dilations and we have $D_r X_j = r^{v_j} X_j$, $j = 1, \dots, n$, on the chosen basis of \mathfrak{g} . The associated group dilations are defined by

$$D_r(x) = r \cdot x := (r^{v_1} x_1, r^{v_2} x_2, \dots, r^{v_n} x_n), \quad x = (x_1, \dots, x_n) \in G, \quad r > 0.$$

In a canonical way this leads to the notions of homogeneity for functions and operators. For instance the degree of homogeneity of x^α and X^α , viewed respectively as a function and a differential operator on G , is

$$[\alpha] = \sum_j v_j \alpha_j.$$

Indeed, let us recall that a vector of \mathfrak{g} defines a left-invariant vector field on G and, more generally, that the universal enveloping Lie algebra of \mathfrak{g} is isomorphic with the left-invariant differential operators; we keep the same notation for the vectors and the corresponding operators.

Recall that a *homogeneous quasi-norm* on G is a continuous function $|\cdot| : G \rightarrow [0, +\infty)$ homogeneous of degree 1 on G which vanishes only at 0. This often replaces the Euclidean norm in the analysis on homogeneous Lie groups:

Proposition 2.1. (1) *Any homogeneous quasi-norm $|\cdot|$ on G satisfies a triangle inequality up to a constant:*

$$\exists C \geq 1 \quad \forall x, y \in G \quad |xy| \leq C(|x| + |y|).$$

It partially satisfies the reverse triangle inequality:

$$\forall b \in (0, 1) \quad \exists C = C_b \geq 1 \quad \forall x, y \in G \quad |y| \leq b|x| \implies ||xy| - |x|| \leq C|y|. \quad (2.2)$$

- (2) Any two homogeneous quasi-norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent in the sense that

$$\exists C > 0 \quad \forall x \in G \quad C^{-1}|x|_2 \leq |x|_1 \leq C|x|_2.$$

An example of a homogeneous quasi-norm is given via

$$|x|_{\nu_o} := \left(\sum_{j=1}^n x_j^{2\nu_o/v_j} \right)^{1/2\nu_o}, \quad (2.3)$$

with ν_o a common multiple to the weights v_1, \dots, v_n .

We will use the L^1 -Young inequality together with the properties of quasi-norms in the following way:

Lemma 2.2. *Let $|\cdot|$ be a quasi-norm and let $s \geq 0$. We set $\omega_s = (1 + |\cdot|)^s$. Let p, q, r be as in Young's inequality in (2.1). Then if f and g are measurable functions, then the following inequality holds (with possibly unbounded quantities):*

$$\|\omega_s f * g\|_{L^r(G)} \leq C \|\omega_s f\|_{L^p(G)} \|\omega_s g\|_{L^q(G)},$$

where the constant C is independent on f, g but may depend on $s, G, |\cdot|$.

Proof. As a quasi-norm satisfies a triangular inequality (see Proposition 2.1), one checks easily

$$\exists C = C_{s,|\cdot|} \quad \forall x, y \in G \quad \omega_s(x) \leq C \omega_s(xy^{-1}) \omega_s(y). \quad (2.4)$$

Thus

$$\begin{aligned} \omega_s(x) |f * g|(x) &\leq \int_G \omega_s(x) |f|(y) |g|(y^{-1}x) dy \\ &\leq \int_G C \omega_s(xy^{-1}) \omega_s(y) |f|(y) |g|(y^{-1}x) dy = C |\omega_s f| * |\omega_s g|. \end{aligned}$$

We conclude with Young's inequality (see (2.1)). \square

Various aspects of analysis on G can be developed in a comparable way with the Euclidean setting sometimes replacing the topological dimension $n = \sum_{\ell=1}^{\infty} \dim \mathfrak{g}_{\ell}$ of the group G by its *homogeneous dimension*

$$Q := \sum_{\ell=1}^{\infty} \ell \dim \mathfrak{g}_{\ell} = v_1 + v_2 + \dots + v_n.$$

For example, there is an analogue of polar coordinates on homogeneous groups with Q replacing n , see [8]:

Proposition 2.3. *Let $|\cdot|$ be a fixed homogeneous quasi-norm on G . Then there is a (unique) positive Borel measure σ on the unit sphere $\mathfrak{S} := \{x \in G : |x| = 1\}$, such that for all $f \in L^1(G)$, we have*

$$\int_G f(x) dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \quad (2.5)$$

This implies the following simple embeddings:

Corollary 2.4. *Let $|\cdot|$ be a fixed homogeneous quasi-norm on G . If $s > Q/2$, then there exists $C > 0$ such that for any measurable function f we have*

$$\|f\|_{L^1(G)} \leq C\|(1 + |\cdot|)^s f\|_{L^2(G)}$$

Moreover as long as $s - \epsilon > Q/2$, there exists $C > 0$ such that for any measurable function f we have

$$\|(1 + |\cdot|)^\epsilon f\|_{L^1(G)} \leq C\|(1 + |\cdot|)^s f\|_{L^2(G)}$$

Proof of Corollary 2.4. By Cauchy-Schwartz' or Hölder's inequality, we have

$$\|(1 + |\cdot|)^\epsilon f\|_{L^1(G)} \leq C_{s,\epsilon}\|(1 + |\cdot|)^s f\|_{L^2(G)},$$

where $C_s := \|(1 + |\cdot|)^{-s+\epsilon}\|_{L^2(G)}$. Using the polar change of coordinates, see Proposition 2.3, C_s is finite for $s - \epsilon > Q/2$. \square

We will need an L^1 -mean value property:

Lemma 2.5. *There exists $C > 0$ such that for any $h \in G$ and any $f \in C^1(G)$ we have*

$$\|f - f(\cdot h)\|_{L^1(G)} \leq C \sum_{\ell=1}^n |h|^{v_\ell} \|X_j f\|_{L^1(G)},$$

and

$$\|f - f(h \cdot)\|_{L^1(G)} \leq C \sum_{\ell=1}^n |h|^{v_\ell} \|\tilde{X}_j f\|_{L^1(G)}.$$

We adapt the argument of [8, Mean Value Theorem 1.33].

Proof of Lemma 2.5. Any $h \in G$ may be written as

$$h = h_1 \dots h_n \quad \text{with} \quad h_\ell := \exp(t_\ell X_\ell) \quad \text{and} \quad |t_\ell| \leq C|h|^{1/v_\ell}.$$

We write

$$\|f - f(h \cdot)\|_{L^1(G)} \leq \sum_{j=1}^n \int_G |f(h_j h_{j+1} \dots h_n x) - f(h_{j+1} \dots h_n x)| dx,$$

and for each term of the sum, we have by the Taylor formula on \mathbb{R} :

$$\begin{aligned} & \int_G |f(h_j h_{j+1} \dots h_n x) - f(h_{j+1} \dots h_n x)| dx \\ &= \int_G \left| \int_{s=0}^{t_\ell} \partial_s f(\exp(s X_j) h_{j+1} \dots h_n x) ds \right| dx \\ &= \int_G \left| \int_{s=0}^{t_\ell} \tilde{X}_j f(\exp(s X_j) h_{j+1} \dots h_n x) ds \right| dx \\ &\leq \int_{s \in [0, t_\ell]} \int_G |\tilde{X}_j f(\exp(s X_j) h_{j+1} \dots h_n x)| dx ds \\ &= \int_{s \in [0, t_\ell]} \int_G |\tilde{X}_j f(y)| dy ds = |t_\ell| \int_G |\tilde{X}_j f(y)| dy. \end{aligned}$$

This shows the right case and the left case is similar. \square

2.2. The dual of G and the Plancherel theorem. Here we set some notation and recall some properties regarding the representations of the group G , especially the Plancherel theorem, and its enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$. The (very) general theory may be found in [4], for a description more adapted to our particular context, see [7].

In this paper, we always assume that the representations of the group G are strongly continuous and acting on separable Hilbert spaces. For a unitary representation π of G , we keep the same notation for the corresponding infinitesimal representation which acts on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra of the group. It is characterised by its action on \mathfrak{g} :

$$\pi(X) = \partial_{t=0} \pi(e^{tX}), \quad X \in \mathfrak{g}. \quad (2.6)$$

The infinitesimal action acts on the space \mathcal{H}_π^∞ of smooth vectors, that is, the space of vectors $v \in \mathcal{H}_\pi$ such that the function $G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$ is of class C^∞ .

The *group Fourier transform* of a function $f \in L^1(G)$ at π is

$$\pi(f) \equiv \widehat{f}(\pi) \equiv \mathcal{F}_G(f)(\pi) = \int_G f(x) \pi(x)^* dx.$$

One checks easily

$$\|\widehat{f}(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq \|f\|_{L^1(G)}. \quad (2.7)$$

We denote by \widehat{G} the set of classes of unitary irreducible representations modulo unitary equivalence, see [4] or [7]. It is a standard Borel space (i.e. a separable complete metrisable topological space equipped with the sigma-algebra generated by the open sets).

From now on, we may identify an unitary irreducible representation with its class in \widehat{G} .

The Plancherel measure is the unique positive Borel standard measure μ on \widehat{G} such that for any $f \in C_c(G)$, we have

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\mathcal{F}_G(f)(\pi)\|_{HS(\mathcal{H}_\pi)}^2 d\mu(\pi).$$

Here $\|\cdot\|_{HS(\mathcal{H}_\pi)}$ denotes the Hilbert-Schmidt norm on \mathcal{H}_π . This implies that the group Fourier transform extends unitarily from $L^1(G) \cap L^2(G)$ to $L^2(G)$ onto

$$L^2(\widehat{G}) := \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* d\mu(\pi),$$

which we identify with the space of μ -square integrable fields on \widehat{G} . The Plancherel formula may be rephrased as

$$\|f\|_{L^2(G)} = \|\widehat{f}\|_{L^2(\widehat{G})}. \quad (2.8)$$

The orbit method furnishes an expression for the Plancherel measure μ , see [3, Section 4.3]. However we will not need this here.

The general theory on locally compact group of type I applies [4]: let $\mathcal{L}(L^2(G))$ be the space of bounded linear operators on $L^2(G)$ and let $\mathcal{L}_L(L^2(G))$ be the subspace of those operators $T \in \mathcal{L}(L^2(G))$ which are left-invariant, that is, commute with the left translation:

$$T(f(g \cdot))(g_1) = (Tf)(gg_1), \quad f \in L^2(G), \quad g, g_1 \in G.$$

Then there exists a field of bounded operators $\widehat{T}(\pi) \in \mathcal{L}(\mathcal{H}_\pi)$, $\pi \in \widehat{G}$, such that

$$\forall f \in L^2(G) \quad \mathcal{F}_G(Tf)(\pi) = \widehat{T}(\pi) \widehat{f}(\pi) \quad \text{for } \mu - \text{almost all } \pi \in \widehat{G}.$$

Moreover the operator norm of T is equal to

$$\|T\|_{\mathcal{L}(L^2(G))} = \sup_{\pi \in \widehat{G}} \|\widehat{T}(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

The supremum here has to be understood as the essential supremum with respect to the Plancherel measure μ . By the Schwartz kernel theorem, any operator $T \in \mathcal{L}_L(L^2(G))$ is a convolution operator and we denote by $T\delta_0 \in \mathcal{S}'(G)$ its convolution kernel: $Tf = f * (T\delta_0)$, $f \in \mathcal{S}(G)$. One may extend the definition of the group Fourier transform to these distributions via $\mathcal{F}_G\{T\delta_0\} = \widehat{T}(\pi)$.

Denoting by $L^\infty(\widehat{G})$ the space of fields of operators $\sigma_\pi \in \mathcal{L}(\mathcal{H}_\pi)$, $\pi \in \widehat{G}$, with

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma_\pi\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty,$$

modulo equivalence under the Plancherel measure μ , we have obtained that $T \in \mathcal{L}(L^2(G))$ implies $\{\mathcal{F}_G\{T\delta_0\} = \widehat{T}(\pi), \pi \in \widehat{G}\} \in L^\infty(\widehat{G})$. Conversely, to any field $\sigma = \{\sigma_\pi, \pi \in \widehat{G}\}$ in $L^\infty(\widehat{G})$, we associate the *Fourier multiplier operator* T_σ via

$$\mathcal{F}_G\{T_\sigma(\phi)\}(\pi) = \sigma_\pi \widehat{\phi}(\pi), \quad \phi \in L^2(G). \quad (2.9)$$

The Plancherel formula implies that $T_\sigma \in \mathcal{L}_L(L^2(G))$ with operator norm bounded by $\|\sigma\|_{L^\infty(\widehat{G})}$. As recalled above, the operator norm is in fact equal to the $L^\infty(\widehat{G})$ -norm of σ . Thus we have obtained the isometric isomorphism of von Neumann algebras

$$\begin{cases} L^\infty(\widehat{G}) & \longrightarrow & \mathcal{L}_L(L^2(G)) \\ \sigma & \longmapsto & T_\sigma \end{cases}$$

with inverse given via $\sigma = \mathcal{F}_G\{T_\sigma\delta_0\}$.

2.3. Rockland operators. Here we recall the definition of Rockland operators and their main properties.

Definition 2.6. A *Rockland operator* \mathcal{R} on G is a left-invariant differential operator which is homogeneous of positive degree and such that for each unitary irreducible non-trivial representation π on G , the operator $\pi(\mathcal{R})$ is injective on \mathcal{H}_π^∞ , that is,

$$\forall v \in \mathcal{H}_\pi^\infty \quad \pi(\mathcal{R})v = 0 \implies v = 0.$$

Although the definition of a Rockland operator would make sense on a homogeneous Lie group (in the sense of [8]), it turns out (cf. [15], see also [5, Lemma 2.2]) that the existence of a (differential) Rockland operator on a homogeneous group implies that the homogeneous group may be assumed to be graded. This explains why we have chosen to restrict our presentation to graded Lie groups. Helffer and Nourrigat proved [9] the Rockland conjecture, that is, that Rockland operators are all the hypoelliptic left-invariant differential operators on a given graded Lie group. Hence Rockland operators may be viewed as an analogue of elliptic operators (with a high degree of homogeneity) in a non-abelian subelliptic context.

Some authors may have different conventions than ours regarding Rockland operators: for instance some choose to consider right-invariant operators and some definitions of a Rockland operator involve only the principal part.

Example 2.7. In the stratified case, one can check easily that any (left-invariant negative) *sub-Laplacian*, that is

$$\mathcal{L} = Z_1^2 + \dots + Z_{n'}^2 \quad \text{with } Z_1, \dots, Z_{n'} \text{ forming any basis of the first stratum } \mathfrak{g}_1, \quad (2.10)$$

is a Rockland operator.

Example 2.8. On any graded group G , it is not difficult to see that the operator

$$\sum_{j=1}^n (-1)^{\frac{\nu_o}{v_j}} c_j X_j^{2\frac{\nu_o}{v_j}} \quad \text{with } c_j > 0, \quad (2.11)$$

is a Rockland operator of homogeneous degree $2\nu_o$ if ν_o is any common multiple of v_1, \dots, v_n .

Hence Rockland operators do exist on any graded Lie group (not necessarily stratified).

If the Rockland operator \mathcal{R} is formally self-adjoint, that is, $\mathcal{R}^* = \mathcal{R}$ as elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, then \mathcal{R} and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and \mathcal{H}_π respectively [8]. We keep the same notation for their self-adjoint extension. We denote by E and E_π their spectral measure:

$$\mathcal{R} = \int_{\mathbb{R}} \lambda dE(\lambda) \quad \text{and} \quad \pi(\mathcal{R}) = \int_{\mathbb{R}} \lambda dE_\pi(\lambda).$$

Example of formally self-adjoint Rockland operators are the positive Rockland operators, that is, Rockland operators \mathcal{R} that satisfy

$$\forall f \in \mathcal{S}(G) \quad \int_G \mathcal{R}f(x) f(x) dx \geq 0.$$

One checks easily that the operator in (2.11) is positive. This shows that positive Rockland operators always exist on any graded Lie group. Note that if G is stratified and \mathcal{L} is a (left-invariant negative) sub-Laplacian, then it is customary to privilege $-\mathcal{L}$ as a positive Rockland operator.

A positive Rockland operator admits a unique self-adjoint extension to an unbounded operator on $L^2(G)$ for which we keep the same notation. The point 0 can be neglected in the spectrum of a positive Rockland operator in the following sense:

Lemma 2.9. *Let \mathcal{R} be a positive Rockland operator with spectral measure E . Then for any $f \in L^2(G)$,*

$$\|E[0, \epsilon)f\|_2 \searrow 0 \quad \text{and} \quad \|E(\epsilon, +\infty)f\|_{L^2(G)} \nearrow \|f\|_{L^2(G)} \quad \text{as } \epsilon \searrow 0.$$

If π is a unitary irreducible representation of G , then the spectrum of $\pi(\mathcal{R})$ is a discrete subset of $(0, \infty)$.

Proof. Let us recall [8] that the heat kernel h_t of \mathcal{R} is by definition the right convolution kernel of $e^{-t\mathcal{R}}$ and that it satisfies $h_t = t^{-\frac{Q}{\nu}} h_1 \circ D_{t^{-\frac{1}{\nu}}}$ with $h_1 \in \mathcal{S}(G)$. This has the two following consequences. Firstly, it yields classically

$$\|e^{-t\mathcal{R}}f\|_2 = \|f * h_t\| \longrightarrow 0, \quad f \in L^2(G)$$

as $t \rightarrow \infty$, which implies the first part of the statement. Secondly, it implies that the operators $\pi(h_t)$, $t > 0$, are compact and form a continuous semi-group. One checks easily that $\pi(\mathcal{R})$ is its infinitesimal generator, and this yields the second part of the statement. See [12]. \square

Consequently, one can define multipliers operators of \mathcal{R} on $(0, +\infty)$, the value of this multiplier function at 0 being negligible. The properties of the functional calculus of \mathcal{R} and of the group Fourier transform imply

Lemma 2.10. *Let \mathcal{R} be a positive Rockland operator of homogeneous degree ν and $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a measurable function. We assume that the domain of the operator $f(\mathcal{R}) = \int_{\mathbb{R}} f(\lambda) dE(\lambda)$ contains $\mathcal{S}(G)$. Then for any $x \in G$,*

$$(f(r^\nu \mathcal{R})\phi) \circ D_r = f(\mathcal{R})(\phi \circ D_r), \quad \phi \in \mathcal{S}(G),$$

where ν denotes the homogeneous degree of \mathcal{R} , and

$$f(r^\nu \mathcal{R})\delta_0(x) = r^{-Q} f(\mathcal{R})\delta_0(r^{-1}x), \quad x \in G, \quad (2.12)$$

where $f(\mathcal{R})\delta_0$ denotes the right convolution kernel of $f(\mathcal{R})$.

Let us recall Hulaniski's Theorem [11]:

Theorem 2.11 (Hulanicki). *Let $|\cdot|$ be a quasi-norm on G , $s \geq 0$, $p \in [1, \infty)$, and $\alpha \in \mathbb{N}_0^n$. Then there exists $C > 0$ and $d \in \mathbb{N}$ such that for any $f \in C^d(0, \infty)$, we have*

$$\int_G (1 + |x|)^s |X^\alpha f(\mathcal{R})\delta_0(x)|^p dx \leq C \sup_{\lambda > 0, \ell=0, \dots, d} (1 + \lambda)^d |\partial_\lambda^{(\ell)} f(\lambda)|,$$

provided that the supremum on the right-hand side is finite.

The same results with the right-invariant vector fields \tilde{X}_j 's instead of the left-invariant vector fields X_j 's hold.

Consequently, if f is a Schwartz function, that is, $f \in \mathcal{S}(\mathbb{R})$ (for instance in $\mathcal{D}(\mathbb{R})$), then $f(\mathcal{R})\delta_0 \in \mathcal{S}(G)$.

We will also use the fact that any two positive Rockland operators are equivalent in the following sense [6]:

Proposition 2.12.

- If \mathcal{R} is a positive Rockland operator, then for any $s \in \mathbb{R}$ the powers \mathcal{R}^s defined by spectral calculus are (unbounded) operators on $L^2(G)$ with domains containing $\mathcal{S}(G)$.

- Let \mathcal{R}_1 and \mathcal{R}_2 be two positive Rockland operators of homogeneous degrees ν_1 and ν_2 respectively. Then $\mathcal{R}_1^{s/\nu_1} \mathcal{R}_2^{-s/\nu_2}$ extends to a bounded operator on $L^2(G)$ for any $s \in \mathbb{R}$:

$$\exists C > 0 \quad \forall \phi \in \mathcal{S}(G) \quad \|\mathcal{R}_1^{s/\nu_1} \phi\|_{L^2(G)} \leq C \|\mathcal{R}_2^{s/\nu_2} \phi\|_{L^2(G)}.$$

Note that Proposition 2.12 implies that if the hypothesis in (1.4) of Theorem 1.1 is satisfied for one positive Rockland operator then it is satisfied for all.

2.4. Difference operators. The difference operators are aimed at replacing the derivatives with respect to the Fourier variable in the Euclidean case.

If q is a continuous function on G , we define Δ_q via

$$\Delta_q \widehat{f}(\pi) = \mathcal{F}_G(qf)(\pi), \quad \pi \in \widehat{G},$$

for any function $f \in \mathcal{D}(G)$. As the group Fourier transform is injective and since $\mathcal{D}(G)$ is dense in $L^p(G)$, $p \in [1, \infty)$, this defines the difference operator Δ_q as a (possibly) unbounded operator on $L^1(G)$ and $L^2(G)$. In particular, for $\alpha \in \mathbb{N}_0^n$, we set

$$\Delta^\alpha := \Delta_{x^\alpha}.$$

Remark 2.13. Difference operators may be defined on more general distributional spaces on G where $\mathcal{D}(G)$ is not necessarily dense, cf [7], for instance on $\mathcal{F}_G^{-1}L^\infty(\widehat{G})$. Then further assumptions are required on q .

The difference operators satisfy the Leibniz rule:

$$\Delta^\alpha(\sigma_1 \sigma_2) = \sum_{[\alpha_1] + [\alpha_2] = [\alpha]} c_{\alpha_1, \alpha_2} \Delta^{\alpha_1} \sigma_1 \Delta^{\alpha_2} \sigma_2, \quad (2.13)$$

where c_{α_1, α_2} are universal constants. By ‘universal constants’, we mean that they depend only on G and the choice of the basis $\{X_j\}_{j=1}^n$.

3. THE SOBOLEV SPACES ON \widehat{G}

The aim of this section is to study Sobolev-type spaces on \widehat{G} defined in the following way:

Definition 3.1. For each $s \geq 0$, we define $H^s(\widehat{G})$ as the space of measurable fields $\sigma = \{\sigma(\pi)\}$ such that $\sigma \in L^2(\widehat{G})$ and $\Delta_{(1+|\cdot|)^s} \sigma \in L^2(\widehat{G})$ where $|\cdot|$ is a quasi-norm on G .

This means that $H^s(\widehat{G})$ is the image via the group Fourier transform of the subspace $L^2(G, (1 + |\cdot|)^{2s})$ of $L^2(G)$:

$$H^s(\widehat{G}) = \mathcal{F}_G \left(L^2(G, (1 + |\cdot|)^{2s}) \right).$$

We will call the spaces $H^s(\widehat{G})$ the Sobolev spaces on \widehat{G} . This vocabulary is justified by the properties stated and proved in this section. We start by showing that the Sobolev spaces on \widehat{G} are Hilbert spaces independent of the quasi-norm:

Proposition 3.2. *Let $s \geq 0$.*

- (1) *The space $H^s(\widehat{G})$ is independent of the quasi-norm $|\cdot|$.*
- (2) *Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators and let $s \geq 0$. The following conditions are equivalent:*
 - $\sigma \in H^s(\widehat{G})$,
 - *there exists a quasi-norm $|\cdot|'$ such that $\mathcal{F}_G^{-1} \sigma \in L^2(G, (1 + |\cdot|')^{2s})$,*
 - $\mathcal{F}_G^{-1} \sigma \in L^2(G, (1 + |\cdot|')^{2s})$ *for any quasi-norm $|\cdot|'$,*

- $\mathcal{F}_G^{-1}\sigma \in L^2(G, \omega_s^2)$ for any continuous function $\omega_s : G \rightarrow (0, \infty)$ equivalent to $(1 + |\cdot|)^s$ in the sense that

$$\exists C > 0 \quad \forall x \in G \quad C^{-1}(1 + |x|)^s \leq \omega_s(x) \leq C(1 + |x|)^s, \quad (3.1)$$

for one (and then all) quasi-norm $|\cdot|$.

- (3) Fixing a weight ω_s satisfying (3.1), the space $H^s(\widehat{G})$ is a Hilbert space when endowed with the sesquilinear form given via

$$\begin{aligned} (\sigma_1, \sigma_2)_{H^s} &:= (\Delta_{\omega_s}\sigma_1, \Delta_{\omega_s}\sigma_2)_{L^2(\widehat{G})} \\ &= \int_{\widehat{G}} \text{Tr}(\Delta_{\omega_s}\sigma_1(\pi) \Delta_{\omega_s}\sigma_2(\pi)^*) d\mu(\pi). \end{aligned}$$

The corresponding norm is given by

$$\|\sigma\|_{H^s, \omega_s} := \|\Delta_{\omega_s}\sigma\|_{L^2(\widehat{G})} = \|\omega_s \mathcal{F}_G^{-1}\sigma\|_{L^2(G)}.$$

Any two weights $\omega_s^{(1)}$ and $\omega_s^{(2)}$ satisfying (3.1) yield equivalent norms on $H^s(\widehat{G})$.

Proof of Proposition 3.2. For any ω_s satisfying (3.1), and any quasi-norm $|\cdot|$, we have $L^2(G, (1 + |\cdot|)^s) = L^2(G, \omega_s)$. If $|\cdot|'$ is another quasi-norm, $(1 + |\cdot|')^s$ is a continuous function satisfying (3.1) since two quasi-norms are equivalent by Proposition 2.1. This together with the isometry $\mathcal{F}_G : L^2(G) \rightarrow L^2(\widehat{G})$ between Hilbert spaces imply the statement. \square

We may allow ourselves to denote the $H^s(\widehat{G})$ -norm by

$$\|\sigma\|_{H^s} := \|\sigma\|_{H^s, \omega_s},$$

when a function ω_s has been fixed.

The space $H^s(\widehat{G})$ is stable by taking the adjoint as one checks easily the following property: if $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ is in $H^s(\widehat{G})$ then $\sigma^* = \{\sigma(\pi)^*, \pi \in \widehat{G}\}$ is also in $H^s(\widehat{G})$ and

$$\|\sigma\|_{H^s, (1+|\cdot|)^s} = \|\sigma^*\|_{H^s, (1+|\cdot|)^s}. \quad (3.2)$$

We have the following inclusions and log-convexity.

Lemma 3.3. *The following continuous inclusions holds for $s_2 \geq s_1 \geq 0$.*

$$L^2(\widehat{G}) = H^0(\widehat{G}) \supset H^{s_1}(\widehat{G}) \supset H^{s_2}(\widehat{G}).$$

If s is between the two non-negative numbers s_1 and s_2 , then

$$\|\sigma\|_{H^s, \omega_s} \leq \|\sigma\|_{H^{s_1}, \omega_{s_1}}^\theta \|\sigma\|_{H^{s_2}, \omega_{s_2}}^{1-\theta},$$

having written $s = \theta s_1 + (1 - \theta)s_2$, with $\theta \in [0, 1]$, and fixed a quasi-norm $|\cdot|$ and $\omega_s = (1 + |\cdot|)^s$.

Proof. The inclusions follow readily from $(1 + |\cdot|)^{s_1} \leq (1 + |\cdot|)^{s_2}$ when $s_2 \geq s_1 \geq 0$. For the log-convexity, we may assume $\theta \neq 0, 1$. Let $\kappa = \mathcal{F}_G^{-1}\sigma \in L^2(\widehat{G})$. We have

$$\begin{aligned} \|\sigma\|_{H^s, \omega_s}^2 &= \|\omega_s \kappa\|_{L^2(G)}^2 = \|(\omega_{s_1} \kappa)^{2\theta} (\omega_{s_2} \kappa)^{2(1-\theta)}\|_{L^1(G)} \\ &\leq \|\omega_{s_1}^{2\theta} \kappa\|_{L^p(G)} \|\omega_{s_2}^{2(1-\theta)} \kappa\|_{L^q(G)}, \end{aligned}$$

by Hölder's inequality with $p = 1/\theta$ and $q = 1/(1 - \theta)$. \square

The difference operators are continuous on the Sobolev spaces:

Lemma 3.4. *Let $s \geq 0$. Let q be a continuous function on G such that $q/\omega_s^{d/s}$ is bounded where $d \geq 0$ and ω_s is a continuous function satisfying (3.1). Then Δ_q maps continuously $H^{s+d}(\widehat{G})$ to $H^s(\widehat{G})$:*

$$\exists C > 0 \quad \forall \sigma \in H^{s+d}(\widehat{G}) \quad \|\Delta_q \sigma\|_{H^s} \leq C \|\sigma\|_{H^{s+d}}.$$

An example of such a function q is any d -homogeneous polynomial. In particular

$$\|\Delta_{x^\alpha} \sigma\|_{H^s} \leq C \|\sigma\|_{H^{s+[\alpha]}}.$$

Proof of Lemma 3.4. We have

$$\begin{aligned} \|\Delta_q \sigma\|_{H^s} &= \|q \omega_s \mathcal{F}_G^{-1} \sigma\|_{L^2(G)} \leq \|q/\omega_s^{d/s}\|_{L^\infty(G)} \|\omega_s^{\frac{d}{s}+1} \mathcal{F}_G^{-1} \sigma\|_{L^2(G)}, \\ &= \|q/\omega_s^{d/s}\|_{L^\infty(G)} \|\sigma\|_{H^{s, \omega'_s}}, \end{aligned}$$

where ω'_s is the continuous function $\omega_s^{\frac{d}{s}+1}$ which satisfies (3.1) with $s' = d + s$. \square

The Sobolev spaces with integer exponents admit an equivalent description:

Lemma 3.5. *Let $s \in \nu_o \mathbb{N}$, that is, s is a positive integer divisible by the dilation weights v_1, \dots, v_n . Then*

$$\sigma \in H^s(\widehat{G}) \iff \forall \alpha \in \mathbb{N}_0^n, [\alpha] \leq s, \quad \Delta_{x^\alpha} \sigma \in L^2(\widehat{G}).$$

Moreover $\sum_{[\alpha] \leq s} \|\Delta_{x^\alpha} \cdot\|_{L^2(\widehat{G})}$ is an equivalent norm on $H^s(\widehat{G})$.

Proof of Lemma 3.5. Let $s \in \nu_o \mathbb{N}$. We consider the quasi-norm is $|\cdot| = |\cdot|_{\nu_o}$ given by (2.3) and the continuous function $\omega_s = (1 + |\cdot|^{2\nu_o})^{\frac{s}{2\nu_o}}$ which satisfies (3.1). Then ω_s^2 is a polynomial in x , and more precisely a linear combination of squared monomials:

$$\omega_s^2(x) = \sum_{[\alpha] \leq s} c_\alpha (x^\alpha)^2$$

for some coefficients (c_α) depending on s and G . Thus

$$\begin{aligned} \|\sigma\|_{H^s, \omega_s}^2 &= \|\omega_s \mathcal{F}_G^{-1} \sigma\|_{L^2(G)}^2 = \int_G \left| \sum_{[\alpha] \leq s} c_\alpha (x^\alpha)^2 \right| |\mathcal{F}_G^{-1} \sigma(x)|^2 dx \\ &\leq \sum_{[\alpha] \leq s} |c_\alpha| \int_G |x^\alpha \mathcal{F}_G^{-1} \sigma(x)|^2 dx \leq C \sum_{[\alpha] \leq s} \|\Delta_{x^\alpha} \sigma\|_{L^2(\widehat{G})}^2. \end{aligned}$$

We have obtained

$$\|\sigma\|_{H^s, \omega_s} \leq C \sum_{[\alpha] \leq s} \|\Delta_{x^\alpha} \sigma\|_{L^2(\widehat{G})}.$$

The reverse inequality follows from Lemma 3.4. \square

Remark 3.6. • In Lemma 3.5, (x^α) may be replaced by any basis of homogeneous polynomials.

- The hypothesis of divisibility of s by v_1, \dots, v_n can not be removed in Lemma 3.5. Indeed let us fix an index $\ell = 1, \dots, n$, and construct a sequence of symbols σ_k , $k \in \mathbb{N}$, via $\mathcal{F}_G^{-1}\sigma_k(x) = 1_{|x_\ell - k| < 1} \prod_{j \neq \ell} 1_{|x_j| < 1}$. One checks easily that

$$\|\sigma_k\|_{H^s} \asymp k^s \quad \text{but} \quad \sum_{[\alpha] \leq s} \|\Delta_{x^\alpha} \sigma_k\|_{L^2(\widehat{G})} \asymp \sum_{[\alpha] \leq s} k^{\alpha_\ell}.$$

If s is a positive integer which is not divisible by v_ℓ then $k^{-s} \sum_{[\alpha] \leq s} k^{\alpha_\ell} \rightarrow 0$ as $k \rightarrow \infty$.

The following analogue of the Sobolev embedding holds as an easy consequence of Corollary 2.4 with (2.7):

Lemma 3.7. *If $\sigma \in H^s(\widehat{G})$ with $s > Q/2$ then $\sigma \in \mathcal{F}_G L^1(G)$ and*

$$\sup_{\pi \in \widehat{G}} \|\sigma\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq \|\mathcal{F}_G^{-1}\sigma\|_{L^1(G)} \leq C \|\sigma\|_{H^s}.$$

As in the Euclidean case, we obtain an algebra for ‘point-wise multiplication’ in the following sense:

Lemma 3.8. *For any σ and τ in $H^s(\widehat{G})$, the product $\sigma\tau = \{\sigma(\pi)\tau(\pi), \pi \in \widehat{G}\}$ satisfies (with possibly unbounded norms)*

$$\|\sigma\tau\|_{H^s} \leq C \left(\|\sigma\|_{H^s} \|\mathcal{F}_G^{-1}\tau\|_{L^1(G)} + \|\mathcal{F}_G^{-1}\sigma\|_{L^1(G)} \|\tau\|_{H^s} \right),$$

with a constant $C > 0$ independent of σ and τ .

Hence for $s > Q/2$, if $\sigma, \tau \in H^s(\widehat{G})$ then $\sigma\tau \in H^s(\widehat{G})$, and $H^s(\widehat{G})$ is a (non-commutative) algebra.

Note that Lemma 2.2 only yields

$$\|\sigma\tau\|_{H^s} \lesssim \|\mathcal{F}_G^{-1}\tau\|_{L^1(\omega_s)} \|\sigma\|_{H^s} \quad (3.3)$$

when a quasi-norm $|\cdot|$ and $\omega_s = (1 + |\cdot|)^s$ with $s \geq 0$ have been fixed. This does not prove Lemma 3.8.

Proof of Lemma 3.8. We fix a quasi-norm $|\cdot|$ and $\omega_s = (1 + |\cdot|)^s$. As a quasi-norm satisfies a triangular inequality (see Proposition 2.1), one checks easily

$$\exists C = C_{s,|\cdot|} \quad \forall x, y \in G \quad \omega_s(x) \leq C (\omega_s(xy^{-1}) + \omega_s(y)). \quad (3.4)$$

Let $\sigma, \tau \in H^s(\widehat{G})$ and $f := \mathcal{F}_G^{-1}\sigma$, $g := \mathcal{F}_G^{-1}\tau$. Then

$$\|\sigma\tau\|_{H^s, \omega_s} = \|\omega_s g * f\|_{L^2(G)}.$$

The inequality in (3.4) implies

$$\omega_s |g * f| \leq C ((\omega_s |g|) * |f| + |g| * (\omega_s |f|)),$$

thus we obtain

$$\begin{aligned} \|\omega_s g * f\|_{L^2(G)} &\leq C (\|(\omega_s |g|) * |f|\|_{L^2(G)} + \||g| * (\omega_s |f|)\|_{L^2(G)}) \\ &\leq C (\|\omega_s |g|\|_{L^2(G)} \|f\|_{L^1(G)} + \|g\|_{L^1(G)} \|\omega_s f\|_{L^2(G)}), \end{aligned}$$

by Young’s inequality (see (2.1)). With Lemma 3.7, the statement follows easily. \square

4. THE MIHLIN-HÖRMANDER CONDITION ON \widehat{G}

In the Euclidean case, the Mihlin-Hörmander condition which implies that a function is an L^p -multiplier for all $p > 1$ is the membership in Sobolev spaces locally uniformly, see the introduction. The aim of this section is to define the membership in Sobolev spaces locally uniformly in our context and express our main multiplier theorem in term of this membership. This requires first to define dilations on \widehat{G} .

4.1. Dilations on \widehat{G} . In this section, we define dilations on the set \widehat{G} . This is possible thanks to the following lemma whose proof is a routine exercise of representation theory for nilpotent Lie groups and the orbit method:

Lemma 4.1. (1) *If π is a unitary irreducible representation of G and $r > 0$ then setting*

$$r \cdot \pi(x) = \pi(rx) \quad , \quad x \in G, \quad (4.1)$$

we have defined a unitary irreducible representation $r \cdot \pi$ of G .

(2) *If π_1 and π_2 are two equivalent unitary irreducible representations of G , then, for any $r > 0$, $r \cdot \pi_1$ and $r \cdot \pi_2$ are two equivalent unitary irreducible representations of G .*

Definition 4.2. For any $\pi \in \widehat{G}$ and any $r > 0$, Equation (4.1) defines a new class

$$r \cdot \pi := D_r(\pi) \in \widehat{G}.$$

These dilations define an action of \mathbb{R}^{*+} on \widehat{G} which interacts nicely with the group structure:

Lemma 4.3. *Let $\pi \in \widehat{G}$ and $r > 0$.*

For any $\alpha \in \mathbb{N}^n$,

$$r \cdot \pi(X^\alpha) = r^{[\alpha]} \pi(X^\alpha).$$

For any positive Rockland operator \mathcal{R} of degree $\nu_{\mathcal{R}}$,

$$r \cdot \pi(\mathcal{R}) = r^{\nu_{\mathcal{R}}} \pi(\mathcal{R})$$

and if $f \in L^\infty(\mathbb{R})$ then (spectral definitions)

$$f(r \cdot \pi(\mathcal{R})) = f(r^{\nu_{\mathcal{R}}} \pi(\mathcal{R})).$$

If $\kappa \in L^2(G) \cup L^1(G)$ then

$$(r \cdot \pi)(\kappa) = \pi(r^{-Q} \kappa(r^{-1} \cdot)).$$

Consequently,

$$\Delta^\alpha \{\widehat{\kappa}(r \cdot \pi)\} = r^{[\alpha]} (\Delta^\alpha \widehat{\kappa})(r \cdot \pi).$$

The proof of Lemma 4.3 is left to the reader.

The Sobolev spaces on \widehat{G} are invariant under these dilations.

Lemma 4.4. *Let $\sigma \in L^2(\widehat{G})$ and $s \geq 0$. Then $\sigma \in H^s(\widehat{G})$ if and only if $\sigma \circ D_r = \{\sigma(r \cdot \pi), \pi \in \widehat{G}\}$ is in $H^s(\widehat{G})$ for one (and then all) $r > 0$.*

Furthermore let us fix a quasi-norm $|\cdot|$ and $\omega_s = (1 + |\cdot|)^s$. For every $r > 0$ and $\sigma \in L^2(\widehat{G})$, we have

$$\|\sigma \circ D_r\|_{H^s, \omega_s} \leq (1+r)^s r^{-\frac{Q}{2}} \|\sigma\|_{H^s, \omega_s}.$$

Proof of Lemma 4.4. Lemma 4.3 and the change of variable D_r yield

$$\|\sigma \circ D_r\|_{H^s, \omega_s} = \|\omega_s r^{-Q} (\mathcal{F}_G^{-1} \sigma) \circ D_{r^{-1}}\|_{L^2} = r^{-\frac{Q}{2}} \|(1 + r|\cdot|)^s \mathcal{F}_G^{-1} \sigma\|_{L^2}.$$

We conclude with $(1 + r|\cdot|)^s \leq (1 + r)^s \omega_s$. \square

4.2. Fields locally uniform in $H^s(\widehat{G})$. The aim of this section is to define and study the Banach space of fields locally uniformly in $H^s(\widehat{G})$. In the Euclidean case, the membership in Sobolev spaces locally uniformly is the Mihlin-Hörmander condition which implies that a function is an L^p -multiplier for all $p > 1$. This motivates the following definition.

Definition 4.5. Let $s \geq 0$. We say that a measurable field of operators $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ is *uniformly locally* in $H^s(\widehat{G})$ on the right, respectively on the left, when there exist a positive Rockland operator \mathcal{R} and a non-zero function $\eta \in \mathcal{D}(0, \infty)$ such that the quantity

$$\|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}} := \sup_{r>0} \|\{\sigma(r \cdot \pi) \eta(\pi(\mathcal{R})), \pi \in \widehat{G}\}\|_{H^s} \quad (4.2)$$

or respectively

$$\|\sigma\|_{H_{l.u.,L}^s, \eta, \mathcal{R}} := \sup_{r>0} \|\{\eta(\pi(\mathcal{R})) \sigma(r \cdot \pi), \pi \in \widehat{G}\}\|_{H^s}, \quad (4.3)$$

is finite.

Our first task will be to show that, as in the Euclidean case, this definition does not depend on the cut-off function. Here we also have to prove that it does not depend on the Rockland operator. This is the object of the following statement which will be proved in Section 5.2.

Proposition 4.6. *Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a measurable field of operators such that $\|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}}$ is finite where \mathcal{R} is a positive Rockland operator and $\eta \in \mathcal{D}(0, \infty)$ a non-zero function. Then for any positive Rockland operator \mathcal{S} and any function $\zeta \in \mathcal{D}(0, \infty)$, the quantity $\|\sigma\|_{H_{l.u.,R}^s, \zeta, \mathcal{S}}$ is finite and there exists a constant $C > 0$ (depending on \mathcal{R}, \mathcal{S} and η, ζ but not on σ) such that*

$$\|\sigma\|_{H_{l.u.,R}^s, \zeta, \mathcal{S}} \leq C \|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}}.$$

We have a similar result for the left case.

We denote by $H_{l.u.,R}^s(\widehat{G})$, resp. $H_{l.u.,L}^s(\widehat{G})$, the space of measurable fields which are uniformly locally in $H^s(\widehat{G})$ on the right, respectively on the left. Furthermore these spaces are Banach spaces with the following properties:

Corollary 4.7. (1) *If $s \geq 0$, the space $H_{l.u.,R}^s(\widehat{G})$ is a Banach space when endowed with any equivalent norm $\|\cdot\|_{H_{l.u.,R}^s, \eta, \mathcal{R}}$, where $\eta \in \mathcal{D}(0, \infty)$ is non-zero and \mathcal{R} is a positive Rockland operator.*
 (2) *We have the continuous inclusion*

$$H_{l.u.,R}^{s_1}(\widehat{G}) \subset H_{l.u.,R}^{s_2}(\widehat{G}), \quad s_1 \geq s_2.$$

(3) If $\sigma \in H_{l.u.,R}^s(\widehat{G})$ and $r_o > 0$, then $\sigma \circ D_{r_o} \in H_{l.u.,R}^s(\widehat{G})$ satisfies

$$\|\sigma \circ D_{r_o}\|_{H_{l.u.,R}^s, \eta, \mathcal{R}} = \|\sigma\|_{H_{l.u.,R}^s, \eta(r_o^{-1} \cdot), \mathcal{R}}.$$

(4) If $s > Q/2$, we have the continuous inclusion of Sobolev type:

$$H_{l.u.,R}^s \subset L^\infty(\widehat{G})$$

We have similar statements for the left case.

This corollary will be proved in Section 5.3. We can already point out that taking the adjoint provides the link between the left and right cases:

Lemma 4.8. *Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators and $s \geq 0$. Then*

$$\sigma \in H_{l.u.,R}^s(\widehat{G}) \iff \sigma^* \in H_{l.u.,L}^s(\widehat{G}),$$

and in this case

$$\|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}} = \|\sigma^*\|_{H_{l.u.,L}^s, \eta, \mathcal{R}}.$$

We can reverse the role of left and right.

Lemma 4.8 follows readily from (3.2).

The following statement gives sufficient conditions for the membership to $H_{l.u.,R}^s(\widehat{G})$ and $H_{l.u.,L}^s(\widehat{G})$.

Proposition 4.9. *Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators and $s \geq 0$. Let \mathcal{R} be a positive Rockland operator and let $\eta \in \mathcal{D}(0, \infty)$ be non-zero.*

(Left): *If $\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma \in L^\infty(\widehat{G})$ for any $|\alpha| \leq N$ for some positive integer $N \in \mathbb{N}$ divisible by v_1, \dots, v_n , then $\sigma \in H_{l.u.,L}^N(\widehat{G})$ and*

$$\|\sigma\|_{H_{l.u.,L}^N, \eta, \mathcal{R}} \leq C \sum_{|\alpha| \leq N} \sup_{\pi \in \widehat{G}} \|\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)},$$

where the constant $C > 0$ does not depend on σ .

(Right): *If $\Delta^\alpha \sigma \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \in L^\infty(\widehat{G})$ for any $|\alpha| \leq N$ for some positive integer $N \in \mathbb{N}$ divisible by v_1, \dots, v_n , then $\sigma \in H_{l.u.,R}^N(\widehat{G})$ and*

$$\|\sigma\|_{H_{l.u.,R}^N, \eta, \mathcal{R}} \leq C \sum_{|\alpha| \leq N} \sup_{\pi \in \widehat{G}} \|\Delta^\alpha \sigma(\pi) \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}}\|_{\mathcal{L}(\mathcal{H}_\pi)},$$

where the constant $C > 0$ does not depend on σ .

Proposition 4.9 will be shown in Section 5.4.

Remark 4.10. The suprema in Proposition 4.9 are independent of a choice of a positive Rockland operator, see Propositions 2.12 and 4.6. Moreover, the condition described in Proposition 4.9 is invariant under dilation by Part (3) of Corollary 4.7 and for the suprema involved in Proposition 4.9, by Lemma 4.3 and the following calculations:

$$\begin{aligned} \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha (\sigma \circ D_{r_o})(\pi) &= r_o^{[\alpha]} \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha (\sigma)(r_o \cdot \pi) \\ &= (r_o \cdot \pi)(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha (\sigma)(r_o \cdot \pi) = \pi_1(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma(\pi_1), \end{aligned}$$

with $\pi_1 = r_o \cdot \pi$. Therefore

$$\sup_{\pi \in \widehat{G}} \|\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha (\sigma \circ D_{r_o})(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} = \sup_{\pi_1 \in \widehat{G}} \|\pi_1(\mathcal{R})^{\frac{[\alpha]}{\nu}} \Delta^\alpha \sigma(\pi_1)\|_{\mathcal{L}(\mathcal{H}_{\pi_1})}. \quad (4.4)$$

4.3. The main result. The main result of this article is the following theorem:

Theorem 4.11. *Let G be a graded nilpotent Lie group. Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators in $L^2(\widehat{G})$.*

If $\sigma \in H_{l.u.,R}^s(\widehat{G}) \cap H_{l.u.,L}^s(\widehat{G})$ for some $s > Q/2$ then the corresponding operator $T = T_\sigma$ is bounded on $L^p(G)$ for any $1 < p < \infty$. Furthermore,

$$\|T\|_{\mathcal{L}(L^p(G))} \leq C \max \left(\|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}}, \|\sigma\|_{H_{l.u.,L}^s, \eta, \mathcal{R}} \right),$$

where $C > 0$ is a constant independent of σ but may depend on p, s, G and a choice of $\eta \in \mathcal{D}(0, \infty)$ and a positive Rockland operator \mathcal{R} .

By Proposition 4.9, Theorem 4.11 implies Theorem 1.1.

The hypotheses and the conclusion of Theorems 4.11 and 1.1 are ‘dilation-invariant’ and do not depend of a choice of a Rockland operator or a function η , see Remark 4.10 and Corollary 4.7.

Theorem 4.11 is proved in Section 5.5 and its proof yields the following more precise version:

Corollary 4.12. *Let G be a graded nilpotent Lie group. Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a μ -measurable field of operators in $L^2(\widehat{G})$ and let T_σ be the corresponding Fourier multiplier operator on $\mathcal{S}(G)$.*

- (1) *If σ is in $H_{l.u.,R}^s$ or $H_{l.u.,L}^s$ for some $s > Q/2$, then T is bounded on $L^2(G)$ with*

$$\|T\|_{\mathcal{L}(L^2(G))} = \sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{op} \leq C_2 \begin{cases} \|\sigma\|_{H_{l.u.,R}^s} \\ \text{or} \\ \|\sigma\|_{H_{l.u.,L}^s} \end{cases},$$

respectively, and C_2 a constant independent of σ .

- (2) *If $\sigma \in H_{l.u.,R}^s$ for some $s > Q/2$ then T is of weak-type L^1 . Moreover there exists a constant $C_1 > 0$ independent of σ , such that*

$$\forall f \in \mathcal{S}(G) \quad \forall \alpha > 0 \quad |\{x : |Tf(x)| > \alpha\}| \leq C_1 \frac{\|\sigma\|_{H_{l.u.,R}^s}}{\alpha} \|f\|_{L^1(G)}.$$

For each $p \in (1, 2)$, there exists a constant $C_p > 0$ independent of σ , such that

$$\forall f \in \mathcal{S}(G) \quad \|Tf\|_{L^p} \leq C_p \|\sigma\|_{H_{l.u.,R}^s} \|f\|_{L^p(G)}.$$

- (3) *If $\sigma \in H_{l.u.,L}^s$ for some $s > Q/2$ then T^* is of weak-type L^1 . Moreover there exists a constant $C_1 > 0$ independent of σ , such that*

$$\forall f \in \mathcal{S}(G) \quad \forall \alpha > 0 \quad |\{x : |T^*f(x)| > \alpha\}| \leq C_1 \frac{\|\sigma\|_{H_{l.u.,L}^s}}{\alpha} \|f\|_{L^1(G)}.$$

For each $p \in (2, \infty)$, there exists a constant $C_p > 0$ independent of σ , such that

$$\forall f \in \mathcal{S}(G) \quad \|Tf\|_{L^p} \leq C_p \|\sigma\|_{H_{l.u.,L}^s} \|f\|_{L^p(G)}.$$

In the statement above, $\|\sigma\|_{H_{l.u.,R}^s}$ denotes a choice of norms $\|\sigma\|_{H_{l.u.,R,\mathcal{R},\eta}^s}$ and similarly of the left case. The constants in the statement depends on this choice.

Remark 4.13. We can apply Corollary 4.12 to obtain properties of Riesz operators for positive Rockland operators. More precisely, let \mathcal{R} be a positive Rockland operator of homogeneous degree ν . We also fix $\alpha \in \mathbb{N}_0^n$. The operator $X^\alpha \mathcal{R}^{-[\alpha]/\nu}$ may be viewed as an operator $\mathcal{S}(G) \rightarrow \mathcal{S}'(G)$ by Proposition 2.12. It is also the Fourier multiplier operator T_σ with multiplier symbol given via

$$\sigma(\pi) = \pi(X)^\alpha \pi(\mathcal{R})^{-[\alpha]/\nu}.$$

We may call this operator a Riesz operator for \mathcal{R} . One checks easily that σ is 0-homogeneous: $\sigma(r \cdot \pi) = \sigma(\pi)$ and that

$$\|\sigma\|_{H_{l.u.,L,\eta,\mathcal{R}}^s} = \|\pi(X)^\alpha \eta_1(\pi(\mathcal{R}))\|_{H^s}.$$

where $\eta_1 \in \mathcal{D}(0, \infty)$ is defined via $\eta_1(\lambda) = \lambda^{-[\alpha]/\nu} \eta(\lambda)$, $\lambda > 0$. By Hulanicki's Theorem, see Theorem 2.11, $\|\pi(X)^\alpha \eta_1(\pi(\mathcal{R}))\|_{H^s}$ is finite. Hence by Corollary 4.12, $\mathcal{R}^{-[\alpha]/\nu} X^\alpha$ admits a unique extension as a bounded operator on $L^p(G)$ for $1 < p \leq 2$. The right condition is much harder to prove.

Similarly or by taking the adjoint, $X^\alpha \mathcal{R}^{-[\alpha]/\nu}$ admits a unique extension as a bounded operator on $L^p(G)$ for $p \geq 2$.

The boundedness of $\mathcal{R}^{-[\alpha]/\nu} X^\alpha$ on $L^p(G)$ for any $p \in (1, \infty)$ has been proved in [6].

5. PROOFS

Here we give the proofs of earlier statements: Proposition 4.6 in Section 5.2, Corollary 4.7 in Section 5.3, Proposition 4.9 in Section 5.4

5.1. Technical lemmata. The proof of Proposition 4.6 will use the following technical lemma:

Lemma 5.1. *Let \mathcal{R}_1 and \mathcal{R}_2 be two positive Rockland operators of homogeneous degrees ν_1 and ν_2 respectively. Let also ζ_1, ζ_2 be two functions in $\mathcal{D}(0, \infty)$. Let also $c, m > 0$ and $s \geq 0$. We fix a quasi-norm $|\cdot|$ and $\omega_s = (1 + |\cdot|)^s$. Then the following sum is finite:*

$$\sum_{j \in \mathbb{Z}} 2^{|j|m} \|\zeta_1(\mathcal{R}_1) \zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)} < \infty.$$

If $\mathcal{R}_1 = \mathcal{R}_2$ then the sum in Lemma 5.1 is finite and the statement is obvious. Lemma 5.1 is easier to prove when \mathcal{R}_1 and \mathcal{R}_2 commute strongly, that is, their spectral measures commute (indeed then one may consider multipliers in both \mathcal{R}_1 and \mathcal{R}_2). But we do not make further hypotheses than the ones in the statement. Instead, we will use the positive Rockland operator $\mathcal{R}_1^{\nu_2} + \mathcal{R}_2^{\nu_1}$.

We will need the following lemma, which is a direct consequence of Hulanicki's theorem (cf. Theorem 2.11) and whose proof is left to the reader.

Lemma 5.2. *Let \mathcal{R} be a positive Rockland operator on G . We fix a quasi-norm $|\cdot|$ on G . Let $\zeta \in \mathcal{D}(0, \infty)$ and $c > 0$. For any $\alpha \in \mathbb{N}_0^n$, there exists $C > 0$ and $d \in \mathbb{N}$*

such that

$$\forall \ell \in \mathbb{Z} \quad \int_G |\tilde{X}^\alpha \zeta(2^{-\ell c} \mathcal{R}) \delta_0(x)| (1 + |x|)^s dx \leq C 2^{d|\ell|}.$$

Proof of Lemma 5.1. Denoting by E_1 , E_2 and E the spectral measures of \mathcal{R}_1 , \mathcal{R}_2 and $\mathcal{R} := \mathcal{R}_1^{\nu_2} + \mathcal{R}_2^{\nu_1}$ respectively, we have for any $a_2, b \geq 0$ that

$$E(-\infty, a_2^{\nu_1}) E_2[a_2, +\infty) \equiv 0 \quad \text{and} \quad E_1(b^{1/\nu_2}, +\infty) E(-\infty, b] \equiv 0. \quad (5.1)$$

Indeed, if $f \in E_2[a_2, +\infty) L^2(G)$ with $a_2 \geq 0$, then

$$(\mathcal{R}f, f)_{L^2(G)} \geq (\mathcal{R}_2^{\nu_1} f, f)_{L^2(G)} \geq a_2^{\nu_1} \|f\|_{L^2(G)}^2, \quad \text{thus} \quad E(-\infty, a_2^{\nu_1}) f = 0.$$

This shows the first equality in (5.1). The second one is proved in a similar way or by taking the adjoint. We also have for $a, b_2 \geq 0$

$$E(a, \infty) E_2(-\infty, b_2] \equiv 0 \quad \text{if } a > (1 + C_o^2) b_2^{\nu_1}, \quad (5.2)$$

where C_o is the finite positive constant (see Proposition 2.12)

$$C_o := \|\mathcal{R}_1^{\frac{\nu_2}{2}} \mathcal{R}_2^{-\frac{\nu_1}{2}}\|_{\mathcal{L}(L^2(G))}.$$

Indeed, if $f \in E_2(-\infty, b_2] L^2(G)$, then

$$(\mathcal{R}_1^{\nu_2} f, f)_{L^2(G)} = \|\mathcal{R}_1^{\frac{\nu_2}{2}} f\|_{L^2(G)}^2 \leq C_o^2 \|\mathcal{R}_2^{\frac{\nu_1}{2}} f\|_{L^2(G)}^2 \leq C_o^2 b_2^{\nu_1} \|f\|_{L^2(G)}^2$$

thus

$$(\mathcal{R}f, f)_{L^2(G)} \leq (C_o^2 + 1) b_2^{\nu_1} \|f\|_{L^2(G)}^2,$$

and (5.2) is proved.

We now fix $a_1, a_2, b_1, b_2 > 0$ such that $\text{supp} \zeta_i \subset (a_i, b_i) \subset (0, \infty)$ for $i = 1, 2$. We also fix a dyadic decomposition, that is, let $\chi \in \mathcal{D}(0, \infty)$ such that $0 \leq \chi \leq 1$, $\text{supp} \chi \subset [1/2, 2]$ and $\sum_{\ell \in \mathbb{Z}} \chi_\ell \equiv 1$ on $(0, \infty)$ where $\chi_\ell(\lambda) := \chi(2^{-\ell} \lambda)$.

Since the point 0 can be neglected in the spectrum of every positive Rockland operator (see Section 2.3), the sum $\sum_{\ell \in \mathbb{Z}} \chi_\ell(\mathcal{R})$ converges to the identity operator in $\mathcal{L}(L^2(G))$. Hence the following equality is justified

$$\|\zeta_1(\mathcal{R}_1) \zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)} \leq \sum_{\ell \in \mathbb{Z}} \|\zeta_1(\mathcal{R}_1) \chi_\ell(\mathcal{R}) \zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)}, \quad (5.3)$$

with a right-hand side possibly unbounded, as $L^1(\omega_s)$ is a Banach space. Each term

$$\|\zeta_1(\mathcal{R}_1) \chi_\ell(\mathcal{R}) \zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)} = 0,$$

vanishes if $2^{\frac{\ell+1}{\nu_1}} < 2^{jc} b_2$ or $a_1^{\nu_2} > 2^{\ell+1}$ or $2^{\ell+1} > (1 + C_o^2)(2^{jc} b_2)^{\nu_1}$ by (5.1) and (5.2). Hence the sum in (5.3) is over ℓ such that $|jc\nu_1 - \ell| \leq M$ and $\ell \geq \ell_0$ for some integers $M \in \mathbb{N}$ and $\ell_0 \in \mathbb{Z}$ independent of $j \in \mathbb{Z}$.

A direct application of Lemmata 2.2 and 5.2 yield

$$\begin{aligned} & \|\zeta_1(\mathcal{R}_1) \chi_\ell(\mathcal{R}) \zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)} \\ & \lesssim \|\zeta_1(\mathcal{R}_1) \delta_0\|_{L^1(\omega_s)} \|\chi_\ell(\mathcal{R}) \delta_0\|_{L^1(\omega_s)} \|\zeta_2(2^{-jc} \mathcal{R}_2) \delta_0\|_{L^1(\omega_s)} \lesssim 2^{|\ell|d_2|j|cd}. \end{aligned}$$

This decay is not enough for our purpose. So we proceed as follows. For each $N \in \mathbb{N}_0$, we set $\tilde{\chi}(\lambda) := \lambda^{-N} \chi(\lambda)$ and we have

$$\chi_\ell(\mathcal{R}) = (2^{-\ell} \mathcal{R})^N \tilde{\chi}_\ell(\mathcal{R}) \quad \text{where} \quad \tilde{\chi}_\ell(\lambda) := \tilde{\chi}(2^{-\ell} \lambda).$$

We observe that

$$\begin{aligned}\zeta_1(\mathcal{R}_1)\chi_\ell(\mathcal{R})\delta_0 &= \zeta_1(\mathcal{R}_1)(2^{-\ell}\mathcal{R})^N\tilde{\chi}_\ell(\mathcal{R})\delta_0 \\ &= 2^{-\ell N}\{\tilde{\chi}_\ell(\mathcal{R})\delta_0\} * \left\{(\tilde{\mathcal{R}}^N)^t\zeta_1(\mathcal{R}_1)\delta_0\right\}.\end{aligned}$$

Hence Lemmata 2.2 and 5.2 yield

$$\|\zeta_1(\mathcal{R}_1)\chi_\ell(\mathcal{R})\zeta_2(2^{-jc}\mathcal{R}_2)\delta_0\|_{L^1(\omega_s)} \lesssim 2^{-\ell N}2^{|\ell|d}2^{|j|cd}.$$

We can now sum over j :

$$\sum_{j \in \mathbb{Z}} 2^{|j|m} \|\zeta_1(\mathcal{R}_1)\zeta_2(2^{-jc}\mathcal{R}_2)\delta_0\|_{L^1(\omega_s)} \lesssim \sum_{\ell \geq \ell_0} \sum_{|jcv_1 - \ell| \leq M} 2^{|j|m} 2^{-\ell N} 2^{|\ell|d} 2^{|j|cd},$$

and this is finite for N chosen large enough. \square

5.2. Proof of Proposition 4.6. Let η and \mathcal{R} be fixed as in Proposition 4.6. We may assume η real valued (otherwise we consider separately $\operatorname{Re} \eta$ and $\operatorname{Im} \eta$). Let $c_o > 0$ such that $2^{c_o}I$ intersects I where I is an open interval inside the support of η . For $\lambda \in \mathbb{R}$ and $j \in \mathbb{Z}$, we set

$$\eta_j(\lambda) = \eta(2^{-c_o j} \lambda) \quad \text{and} \quad \alpha(\lambda) := \sum_{j \in \mathbb{Z}} \eta_j^2(\lambda).$$

Since the support of η is compact and in $(0, \infty)$, if $\lambda \leq 0$ then $\alpha(\lambda) = 0$ and if $\lambda > 0$, the sum giving $\alpha(\lambda)$ is finite and even uniformly locally finite on $(0, \infty)$. As η is smooth, α is also smooth. The choice of c_o implies that α is positive on $(0, \infty)$. The functions η/α and η^2/α are in $\mathcal{D}(0, \infty)$. We have

$$\forall \lambda \in \mathbb{R}, j \in \mathbb{Z} \quad \alpha(2^{jc_o} \lambda) = \alpha(\lambda),$$

thus

$$\forall \lambda > 0 \quad \sum_{j \in \mathbb{Z}} \frac{\eta_j^2}{\alpha}(2^{-jc_o} \lambda) = \sum_{j \in \mathbb{Z}} \frac{\eta_j^2}{\alpha}(\lambda) = 1.$$

Since the point 0 can be neglected in the spectrum of every positive Rockland operator (see Section 2.3), we have

$$\mathbf{I}_{\mathcal{L}(L^2(G))} = \sum_{j \in \mathbb{Z}} \frac{\eta_j^2}{\alpha}(\mathcal{R}) \quad \text{and} \quad \mathbf{I}_{\mathcal{H}_\pi} = \sum_{j \in \mathbb{Z}} \frac{\eta_j^2}{\alpha}(\pi(\mathcal{R})). \quad (5.4)$$

as sums converging in $\mathcal{L}(L^2(G))$ and in $\mathcal{L}(\mathcal{H}_\pi)$, respectively.

Inserting the above sum, we obtain

$$\|\sigma(r \cdot \pi)\zeta(\pi(\mathcal{S}))\|_{H^s} \leq \sum_{j \in \mathbb{Z}} \|\sigma(r \cdot \pi) \frac{\eta_j^2}{\alpha}(\pi(\mathcal{R}))\zeta(\pi(\mathcal{S}))\|_{H^s}.$$

If we set $\pi' = 2^{-\frac{jc_o}{\nu_{\mathcal{R}}}} \cdot \pi$ for each $j \in \mathbb{Z}$, Lemma 4.3 yields

$$\sigma(r \cdot \pi) \frac{\eta_j^2}{\alpha}(\pi(\mathcal{R}))\zeta(\pi(\mathcal{S})) = \sigma((r2^{\frac{jc_o}{\nu_{\mathcal{R}}}}) \cdot \pi') \frac{\eta_j^2}{\alpha}(\pi'(\mathcal{R}))\zeta(2^{-jc} \cdot \pi'(\mathcal{S})),$$

where $c := c_o \nu_S / \nu_R$. We fix a quasi-norm $|\cdot|$ and $\omega_s := (1 + |\cdot|)^s$. By Lemma 4.4, we have

$$\begin{aligned} & \|\sigma(r \cdot \pi) \frac{\eta_j^2}{\alpha} (\pi(\mathcal{R})) \zeta(\pi(\mathcal{S}))\|_{H^s} \\ & \leq (1 + 2^{-\frac{ic_o}{\nu_R}})^s 2^{\frac{ic_o}{\nu_R} \frac{Q}{2}} \|\sigma((r 2^{\frac{ic_o}{\nu_R}}) \cdot \pi') \frac{\eta^2}{\alpha} (\pi'(\mathcal{R})) \zeta(2^{-jc} \cdot \pi'(\mathcal{S}))\|_{H^s} \\ & \lesssim 2^{|j|m} \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta} \left\| \mathcal{F}_G^{-1} \left\{ \frac{\eta}{\alpha} (\pi(\mathcal{R})) \zeta(2^{-jc} \cdot \pi(\mathcal{S})) \right\} \right\|_{L^1(\omega_s)} \end{aligned}$$

having used (3.3) for the H^s -norm and set $m := \frac{c_o}{\nu_R} (s + \frac{Q}{2})$. We have obtained

$$\|\sigma(r \cdot \pi) \zeta(\pi(\mathcal{S}))\|_{H^s} \lesssim \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta} \sum_{j \in \mathbb{Z}} 2^{|j|m} \left\| \mathcal{F}_G^{-1} \left\{ \frac{\eta}{\alpha} (\pi(\mathcal{R})) \zeta(2^{-jc} \cdot \pi(\mathcal{S})) \right\} \right\|_{L^1(\omega_s)}$$

and this sum is bounded by Lemma 5.1. We conclude the proof of Proposition 4.6 by taking the supremum over $r > 0$ on the left-hand side.

5.3. Proof of Corollary 4.7. If $\|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}} = 0$, then by Lemma 4.4,

$$\|\sigma \eta(r \pi(\mathcal{R}))\|_{H^s} = 0,$$

and the field $\sigma(\pi) \eta(r \pi(\mathcal{R}))$ is identically zero for any $r > 0$, since $H^s(\widehat{G})$ is a normed space. Choosing η such that e.g. $\eta \equiv 1$ on $[1, 2]$, this implies that for any $a, b \geq 0$, $\sigma(\pi) E_\pi[a, b] \equiv 0$ where E_π is the spectral resolution of $\pi(\mathcal{R})$ (or equivalently the group Fourier transform of the spectral resolution of \mathcal{R} , see [7]). Hence $\sigma = 0$.

Let $\{\sigma_\ell\}$ be a Cauchy sequence in $H_{l.u.,R}^s(\widehat{G})$, that is,

$$\forall \epsilon > 0 \quad \exists \ell_\epsilon \in \mathbb{N} : \forall \ell_1, \ell_2 \geq \ell_\epsilon \quad \forall r > 0 \quad \|(\sigma_{\ell_1} - \sigma_{\ell_2})(r \cdot \pi) \eta(\pi(\mathcal{R}))\|_{H^s} \leq \epsilon. \quad (5.5)$$

This implies that $\{\sigma_\ell(r \cdot \pi) \eta(\pi(\mathcal{R}))\}$ is a Cauchy sequence in the Banach space $H^s(\widehat{G})$ for each $r > 0$ fixed. For the same reasons as above, this shows that $\{\sigma_\ell E_\pi[a, b]\}$ is a Cauchy sequence in the Banach space $H^s(\widehat{G})$. Hence it converges towards a limit $\sigma^{([a,b])}$ in $H^s(\widehat{G})$ with $\sigma^{([a,b])} = \sigma^{([c,d])} E_\pi[a, b]$ if $[a, b] \subset [c, d]$. This defines a field of operators σ which satisfies for each $r > 0$ fixed:

$$\lim_{\ell \rightarrow \infty} \sigma_\ell(r \cdot \pi) \eta(\pi(\mathcal{R})) = \sigma(r \cdot \pi) \eta(\pi(\mathcal{R})).$$

Passing to the limit in (5.5), this shows that σ is also the limit of $\{\sigma_\ell\}$ in $H_{l.u.,R}^s(\widehat{G})$. This shows Part (1) of Corollary 4.7.

Part (2) follows from the similar inclusions for $H^s(\widehat{G})$, see Lemma 3.3. Part (3) is easily checked.

It remains to show Part (4). Let $s > Q/2$. We may choose $\eta \in \mathcal{D}(0, \infty)$ supported in $[1/2, 4]$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $[1, 2]$. The properties of the functional calculus yield

$$\begin{aligned} & \sup_{\pi \in \widehat{G}} \|\sigma(\pi) E_\pi[r^{-1}, 2r^{-1}]\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq \sup_{\pi \in \widehat{G}} \|\sigma(r \cdot \pi) E[1, 2] \eta(\pi(\mathcal{R}))\|_{\mathcal{L}(\mathcal{H}_\pi)} \\ & \leq \sup_{\pi \in \widehat{G}} \|\sigma(r \cdot \pi) \eta(\pi(\mathcal{R}))\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq C \|\sigma\|_{H_{l.u.,R}^s, \eta, \mathcal{R}} \end{aligned}$$

by the Sobolev embedding of $H^s(\widehat{G})$, see Lemma 3.7. Here, the constant C is independent of $r > 0$, thus the supremum over $r > 0$ of $\sup_{\pi \in \widehat{G}} \|\sigma(\pi)E_\pi[r^{-1}, 2r^{-1}]\|_{\mathcal{L}(\mathcal{H}_\pi)}$ is finite. This shows that $\sigma \in L^\infty(\widehat{G})$. This also concludes the proof of Corollary 4.7.

5.4. Proof of Proposition 4.9. We will prove the second statement for the right spaces. Let $N \in \nu_o \mathbb{N}$, that is, a positive integer divisible by ν_1, \dots, ν_n . Let $\sigma \in L^2(\widehat{G})$ be such that $\Delta^\alpha \sigma \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \in L^2(\widehat{G})$ for any $|\alpha| \leq N$. By Lemma 3.5, we have

$$\begin{aligned} \|\sigma(r \cdot \pi)\eta(\pi(\mathcal{R}))\|_{H^N(\widehat{G})} &\asymp \sum_{|\alpha| \leq N} \|\Delta_{x^\alpha}(\sigma(r \cdot \pi)\eta(\pi(\mathcal{R})))\|_{L^2(\widehat{G})} \\ &\lesssim \sum_{|\alpha_1| + |\alpha_2| \leq N} \|\Delta_{x^{\alpha_1}}\sigma(r \cdot \pi) \Delta_{x^{\alpha_2}}\eta(\pi(\mathcal{R}))\|_{L^2(\widehat{G})}, \end{aligned}$$

by the Leibniz formula, see (2.13). Inserting powers of $\pi(\mathcal{R})$, we have for each term above the estimate

$$\begin{aligned} &\|\Delta_{x^{\alpha_1}}\sigma(r \cdot \pi) \Delta_{x^{\alpha_2}}\eta(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} \\ &= \|\Delta_{x^{\alpha_1}}\sigma(r \cdot \pi) \pi(\mathcal{R})^{\frac{[\alpha_1]}{\nu}} \pi(\mathcal{R})^{-\frac{[\alpha_1]}{\nu}} \Delta_{x^{\alpha_2}}\eta(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} \\ &\leq \|\Delta_{x^{\alpha_1}}\sigma(r \cdot \pi) \pi(\mathcal{R})^{\frac{[\alpha_1]}{\nu}}\|_{L^\infty(\widehat{G})} \|\pi(\mathcal{R})^{-\frac{[\alpha_1]}{\nu}} \Delta_{x^{\alpha_2}}\eta(\pi(\mathcal{R}))\|_{L^2(\widehat{G})}. \end{aligned}$$

By (4.4), we have

$$\|\Delta_{x^{\alpha_1}}\sigma(r \cdot \pi) \pi(\mathcal{R})^{\frac{[\alpha_1]}{\nu}}\|_{L^\infty(\widehat{G})} = \sup_{\pi_1 \in \widehat{G}} \|\Delta^\alpha \sigma(\pi_1) \pi_1(\mathcal{R})^{\frac{[\alpha]}{\nu}}\|_{\mathcal{L}(\mathcal{H}_{\pi_1})}.$$

By Hulanicki's Theorem, see Theorem 2.11, the function $g_{\alpha_2} := x^{\alpha_2} \eta(\pi(\mathcal{R}))\delta_0$ is Schwartz, therefore it is in the domain of the operator $\mathcal{R}^{-\frac{[\alpha_1]}{\nu}}$, see Proposition 2.12. Thus the quantity

$$\|\pi(\mathcal{R})^{-\frac{[\alpha_1]}{\nu}} \Delta_{x^{\alpha_2}}\eta(\pi(\mathcal{R}))\|_{L^2(\widehat{G})} = \|\mathcal{R}^{-\frac{[\alpha_1]}{\nu}} g_{\alpha_2}\|_{L^2(G)}$$

may be viewed as a finite constant. We have therefore obtained that

$$\|\sigma(r \cdot \pi)\eta(\pi(\mathcal{R}))\|_{H^N(\widehat{G})} \lesssim \sum_{|\alpha_1| \leq N} \sup_{\pi_1 \in \widehat{G}} \|\Delta^\alpha \sigma(\pi_1) \pi_1(\mathcal{R})^{\frac{[\alpha]}{\nu}}\|_{\mathcal{L}(\mathcal{H}_{\pi_1})}.$$

Taking the supremum over r on the left hand side proves Proposition 4.9 for the condition on the right. For the condition on the left, one can proceed in a similar way or obtain it by taking the adjoint from the condition on the right, see Lemma 4.8.

5.5. Proof of Theorem 4.11. Let $\sigma \in H_{l.u., R}^s$ with $s > Q/2$. We want to show that the Fourier multiplier operator T_σ admits an L^p -bounded extension and the classical way to do this is to prove that T_σ is a Calderón-Zygmund operator on the space of homogeneous type G , see [2, Ch. III].

Let $\eta \in \mathcal{D}(0, \infty)$ be supported in $[1/2, 2]$, valued in $[0, 1]$ and satisfying $\sum_{j \in \mathbb{Z}} \eta_j \equiv 1$ on $(0, \infty)$ where $\eta_j(\lambda) = \eta(2^{-j}\lambda)$. For each $j \in \mathbb{Z}$ and $\pi \in \widehat{G}$, we set

$$\sigma_j(\pi) = \sigma(2^{-j} \cdot \pi)\eta(\pi(\mathcal{R})).$$

We have $\sigma_j \in H^s(\widehat{G})$ with

$$\|\sigma_j\|_{H^s} \leq \|\sigma\|_{H_{l,u.,R}^s, \mathcal{R}, \eta}. \quad (5.6)$$

By Corollary 4.7 (4), σ and the σ_j 's are in $L^\infty(\widehat{G})$ and thus define Fourier multipliers

$$T : \phi \mapsto \mathcal{F}_G^{-1}\{\sigma\widehat{\phi}\} \quad \text{and} \quad T_j : \phi \mapsto \mathcal{F}_G^{-1}\{\sigma_j\widehat{\phi}\},$$

which are bounded on $L^2(G)$. Their convolution kernels are respectively $\kappa := \mathcal{F}_G^{-1}\sigma \in \mathcal{S}'(G)$, and $\kappa_j := \mathcal{F}_G^{-1}\sigma_j$ is in $L^2(G)$.

By Lemma 3.7, the function κ_j is integrable.

Remark 5.3. Even if it is not needed we can easily show that

$$\int_G \kappa_j(x) dx = 0.$$

Indeed denoting by $1_{\widehat{G}}$ the trivial representation, we have

$$\int_G \kappa_j(x) dx = \widehat{\kappa}_j(1_{\widehat{G}}) = \sigma_j(1_{\widehat{G}}) = \sigma(2^{-j} \cdot 1_{\widehat{G}}) \eta(1_{\widehat{G}}(\mathcal{R})).$$

Since the infinitesimal representation of $1_{\widehat{G}}$ is identically zero and η is supported away from 0, we have $\eta(1_{\widehat{G}}(\mathcal{R})) = 0$ and therefore the integral of κ_j is zero.

The sum $\sum_{j \in \mathbb{Z}} \eta_j(\mathcal{R})$ converge towards the identity operator in $\mathcal{L}(L^2(G))$. Formally we have

$$T = \sum_{j \in \mathbb{Z}} T_j \eta_j(\mathcal{R}), \quad \sigma = \sum_{j \in \mathbb{Z}} \sigma_j(2^j \pi), \quad \text{and} \quad \kappa = \sum_{j \in \mathbb{Z}} 2^{-Qj} \kappa_j(2^{-j}).$$

Let us prove that the last sum has a meaning and that the first Calderón-Zygmund condition is satisfied:

Lemma 5.4. *The function κ is locally integrable on $G \setminus \{0\}$. Moreover the sum $\sum_{j \in \mathbb{Z}} 2^{-Qj} \kappa_j(2^{-j} \cdot)$ converges to κ in $L_{loc}^1(G \setminus \{0\})$.*

Proof. Let $m \in \mathbb{Z}$ be fixed. By the change of variable given by the dilation D_j , we have for each $j \in \mathbb{Z}$ that

$$\int_{2^m \leq |x| \leq 2^{m+1}} |2^{-Qj} \kappa_j(2^{-j} x)| dx = \int_{2^{m-j} \leq |x| \leq 2^{m-j+1}} |\kappa_j(x)| dx.$$

If $m - j \geq 0$, we have

$$\begin{aligned} \int_{2^{m-j} \leq |x| \leq 2^{m-j+1}} |\kappa_j(x)| dx &= \int_{2^{m-j} \leq |x| \leq 2^{m-j+1}} |\kappa_j(x)| (1 + |x|)^\epsilon (1 + |x|)^{-\epsilon} dx \\ &\lesssim 2^{(m-j)(-\epsilon)} \|\kappa_j(1 + |\cdot|)^\epsilon\|_{L^1(G)} \lesssim 2^{(j-m)\epsilon} \|\kappa_j(1 + |\cdot|)^s\|_{L^2(G)}, \end{aligned}$$

by Corollary 2.4, as long as $s - \epsilon > Q/2$.

If $m - j < 0$, we have by the Cauchy-Schwartz inequality that

$$\begin{aligned} \int_{2^{m-j} \leq |x| \leq 2^{m-j+1}} |\kappa_j(x)| dx &= \int_G (1 + |x|)^s |\kappa_j(x)| (1 + |x|)^{-s} 1_{2^{m-j} \leq |x| \leq 2^{m-j+1}} dx \\ &\leq \|\kappa_j(1 + |\cdot|)^s\|_{L^2(G)} \|(1 + |\cdot|)^{-s} 1_{2^{m-j} \leq |x| \leq 2^{m-j+1}}\|_{L^2(G)}. \end{aligned}$$

Note that

$$\|(1 + |\cdot|)^{-s} 1_{2^{m-j} \leq |x| \leq 2^{m-j+1}}\|_{L^2(G)} \lesssim 2^{(m-j)\frac{Q}{2}},$$

and that by (5.6),

$$\|\kappa_j(1 + |\cdot|)^s\|_{L^2(G)} = \|\sigma_j\|_{H^s} \leq \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta}.$$

We choose $\epsilon = (s + Q/2)/2$. We can now sum over $j \in \mathbb{Z}$ to obtain

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \int_{2^m \leq |x| \leq 2^{m+1}} |2^{-Qj} \kappa_j(2^{-j}x)| dx &= \sum_{j=-\infty}^{m-1} + \sum_{j=m}^{\infty} \\ &\lesssim \sum_{j=-\infty}^{m-1} 2^{\frac{Q}{2}(m-j)} \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta} + \sum_{j=m}^{\infty} 2^{(j-m)\epsilon} \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta} \\ &\lesssim \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta}. \end{aligned}$$

This implies that $\kappa = \sum_{j \in \mathbb{Z}} 2^{-Qj} \kappa_j(2^{-j}\cdot)$ is integrable on $\{2^m \leq |x| \leq 2^{m+1}\}$. Therefore κ is locally integrable on $G \setminus \{0\}$. \square

Let us show the Calderón-Zygmund inequality on the kernel:

Lemma 5.5. *Let us rewrite*

$$K(x, y) = \kappa(y^{-1}x) \quad \text{and} \quad d(x, y) = |y^{-1}x|.$$

There exists $C > 0$ such that for any two distinct points $y, y' \in G$ we have:

$$\int_{d(x,y) > 4cd(y,y')} |K(x, y) - K(x, y')| dx \leq C \|\sigma\|_{H_{l.u.,R}^s, \mathcal{R}, \eta}.$$

For $K_(x, y) = \kappa^*(y^{-1}x) = \bar{\kappa}(x^{-1}y)$, there exists $C > 0$ such that for any two distinct points $y, y' \in G$ we have*

$$\int_{d(x,y) > 4cd(y,y')} |K_*(x, y) - K_*(x, y')| dx \leq C \|\sigma\|_{H_{l.u.,L}^s, \mathcal{R}, \eta}.$$

Here c denotes the constant in the triangular inequality for the chosen quasi-norm, see Proposition 2.1.

Proof of Lemma 5.5. Let $y, y' \in G$ be two distinct points. Let h be the point $h := y'^{-1}y$ in $G \setminus \{0\}$ and let $m \in \mathbb{Z}$ be the integer such that $2^m \leq 4c|h| < 2^{m+1}$. After the change of variable $z = y^{-1}x$ we see that

$$\int_{d(x,y) > 4cd(y,y')} |K(x, y) - K(x, y')| dx = \int_{|z| > 4c|h|} |\kappa(z) - \kappa(hz)| dz \leq \sum_{j \in \mathbb{Z}} I_j,$$

where

$$I_j := \int_{|z| > 4c|h|} |2^{-jQ} \kappa_j(2^{-j}z) - 2^{-jQ} \kappa_j(2^{-j}(hz))| dz,$$

since $\kappa = \sum_j 2^{-jQ} \kappa(2^{-j}\cdot)$. Using the change of variable $2^{-j}z = w$, we have

$$I_j = \int_{2^j|w| > 4c|h|} |\kappa_j(w) - \kappa_j((2^{-j}h)w))| dw.$$

If $j < m$ we use

$$I_j \leq \int_{2^j|w| > 4c|h|} |\kappa_j(w)| dw + \int_{2^j|(2^{-j}h)^{-1}w'| > 4c|h|} |\kappa_j(w')| dw',$$

after the change of variable $w' = (2^{-j}h)w$. The triangular inequality implies that

$$2^j |(2^{-j}h)^{-1}w'| > 4c|h| \implies |w'| > 3c2^{-j}|h| \geq \frac{3}{4}2^{-j+m}.$$

Therefore,

$$I_j \leq 2 \int_{|w| > \frac{3}{4}2^{-j+m}} |\kappa_j(w)| dw \lesssim 2^{\epsilon(j-m)} \|(1 + |\cdot|)^\epsilon \kappa_j\|_{L^1(G)}.$$

By Corollary 2.4 and Lemma 3.7 with (5.6), we have

$$\|(1 + |\cdot|)^\epsilon \kappa_j\|_{L^1(G)} \lesssim \|\sigma\|_{H_{l.u.}^s, \mathcal{R}, \eta}.$$

So we have obtained in the case $j < m$,

$$I_j \lesssim 2^{\epsilon(j-m)} \|\sigma\|_{H_{l.u.}^s, \mathcal{R}, \eta}.$$

If $j \geq m$, we use the L^1 -mean value theorem given in Lemma 2.5:

$$I_j \lesssim \sum_{\ell=1}^n |2^{-j}h|^{v_\ell} \|\tilde{X}_\ell \kappa_j\|_{L^1(G)} \lesssim 2^{m-j} \sum_{\ell=1}^n \|\tilde{X}_\ell \kappa_j\|_{L^1(G)}.$$

as $1 \leq v_1 \leq \dots \leq v_n$. By Corollary 2.4 and Lemma 3.7, we have

$$\|\tilde{X}_\ell \kappa_j\|_{L^1(G)} \lesssim \|(\tilde{X}_\ell \kappa_j)(1 + |\cdot|)^s\|_{L^2(G)}$$

and by the Plancherel formula (see (2.8))

$$\begin{aligned} \|(\tilde{X}_\ell \kappa_j)(1 + |\cdot|)^s\|_{L^2(G)} &= \|\sigma_j(\pi)\pi(X_\ell)\|_{H^s(\hat{G})} \\ &= \|\sigma(2^{-j} \cdot \pi)\eta(\pi(\mathcal{R})) \omega(\pi(\mathcal{R}))\pi(X_\ell)\|_{H^s(\hat{G})}, \end{aligned}$$

where $\omega \in \mathcal{D}(0, \infty)$ is identically equal to 1 on the support of η . By Hulanicki's Theorem, cf. Theorem 2.11, the function $g_\ell := \mathcal{F}_G^{-1}\{\omega(\pi(\mathcal{R}))\pi(X_\ell)\}$ is Schwartz. By (3.3), we have

$$\begin{aligned} \|\sigma(2^{-j} \cdot \pi)\eta(\pi(\mathcal{R})) \omega(\pi(\mathcal{R}))\pi(X_\ell)\|_{H^s(\hat{G})} &= \|\sigma(2^{-j} \cdot \pi)\eta(\pi(\mathcal{R})) \hat{g}_\ell(\pi)\|_{H^s(\hat{G})} \\ &\lesssim \|\sigma(2^{-j} \cdot \pi)\eta(\pi(\mathcal{R}))\|_{H^s(\hat{G})} \leq \|\sigma\|_{H_{l.u., R}^s, \mathcal{R}, \eta}. \end{aligned}$$

So we have obtained in the case $j \geq m$ that

$$I_j \lesssim 2^{m-j} \|\sigma\|_{H_{l.u., R}^s, \mathcal{R}, \eta}.$$

We can now go back to

$$\sum_{j \in \mathbb{Z}} I_j \lesssim \sum_{j < m} 2^{\epsilon(j-m)} \|\sigma\|_{H_{l.u., R}^s, \mathcal{R}, \eta} + \sum_{j \geq m} 2^{m-j} \|\sigma\|_{H_{l.u., R}^s, \mathcal{R}, \eta} \lesssim \|\sigma\|_{H_{l.u., R}^s, \mathcal{R}, \eta}.$$

For K_* , after the change of variable $z = x^{-1}y$ and setting $h' = y^{-1}y'$, we see that

$$\int_{d(x, y) > 4cd(y, y')} |K_*(x, y) - K_*(x, y')| dx = \int_{|z| > 4c|h|} |\kappa(z) - \kappa(zh')| dz.$$

We proceed exactly in the same way as above using left invariant vector fields X_ℓ . \square

Hence the operator T satisfies the hypotheses of the Calderón-Zygmund theorem in the context of graded Lie groups, and more generally on spaces of homogeneous type cf. [2, Ch. III]. This implies Theorem 4.11 and the following proof of Corollary 4.12.

Proof of Corollary 4.12. Part (1) follows from Corollary 4.7. For Part (2), Lemmata 5.4 and 5.5 show that, if $\sigma \in H_{l.u.,R}^s$ for some $s > Q/2$, then κ is a Calderón-Zygmund kernel, see [2, Ch. III]. We proceed in the same way for Part (3): using Lemma 4.8: if $\sigma \in H_{l.u.,L}^s$ for some $s > Q/2$, then κ^* is a Calderón-Zygmund kernel. As $T^* = T_{\sigma^*}$, this shows (3). \square

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